Using Ricci Flow to Improve Your Manifold's Shape (and to Prove the Poincaré Conjecture)

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April 4, 2007
PLU Math Seminar

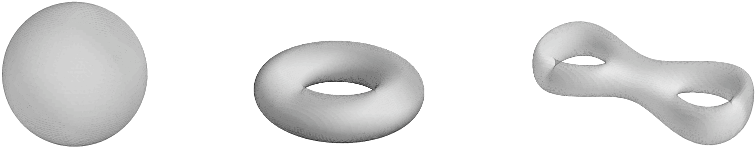
Outline

Disclaimers

- this talk with be filled with white lies and half truths
- the speaker has almost no understanding of the technical details in recent work on Ricci flow

Poincaré Conjecture

- Poincaré Conjecture: The sphere is the only three dimensional closed manifold that is simply connected.
- terminology
  - manifold: a space that locally looks like Euclidean space
  - closed: finite in extent with no edges
  - simply connected: every closed loop can be shrunk to a point or, equivalently, every circle is the boundary of a disk
- examples of closed manifolds (two-dimensional): sphere, torus, “two-holed” torus

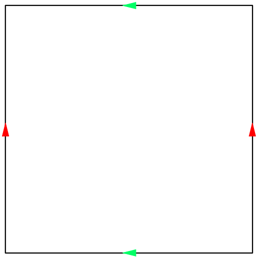


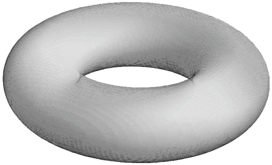
- topology looks at shape while ignoring information about distances and angles
- for geometry, require manifolds to be *smooth* and *orientable*

Outline

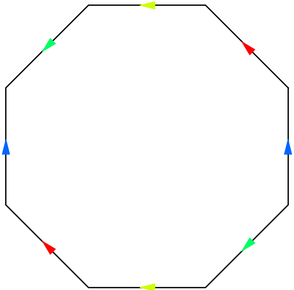
Gluing to get surfaces

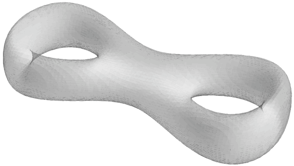
- Glue square to get torus





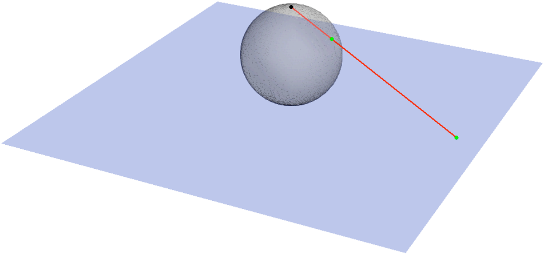
- Glue octagon to get two-holed torus

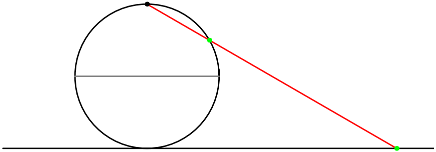




Gluing to get surfaces

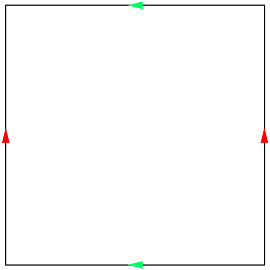
- Glue all points at infinity to get sphere using stereographic projection





A small taste of geometry

- walk around one point on the torus



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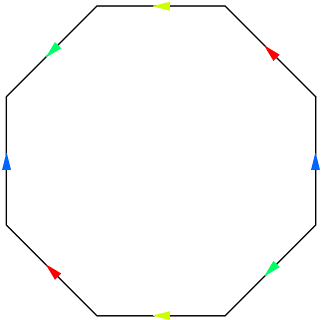

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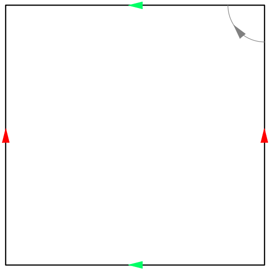

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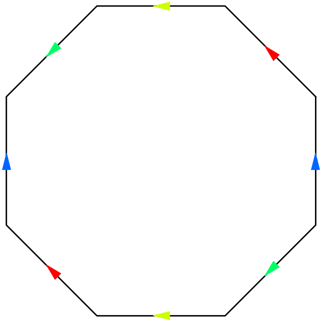

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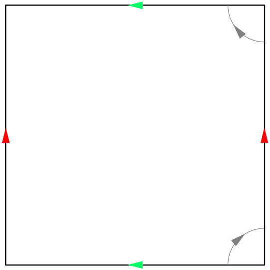

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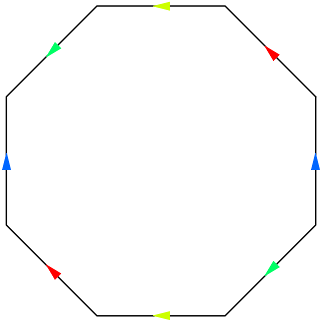

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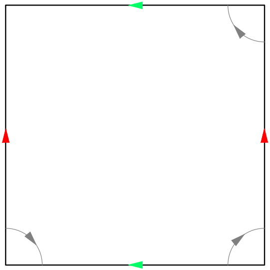

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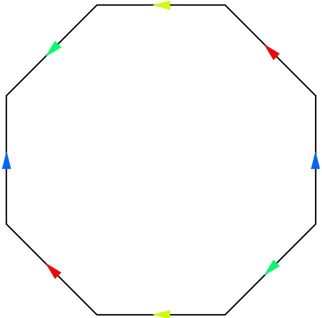

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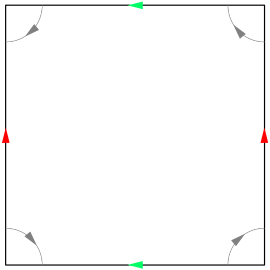

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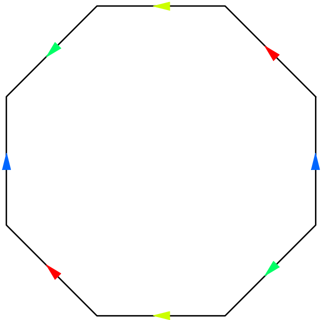

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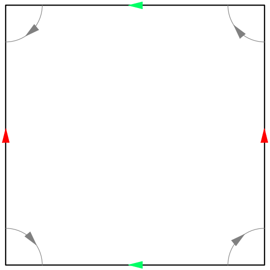

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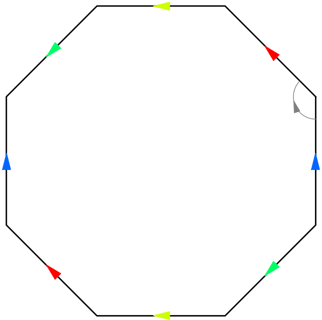

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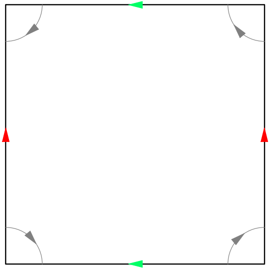

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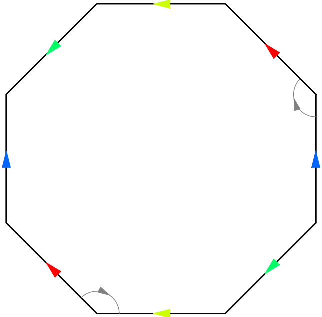

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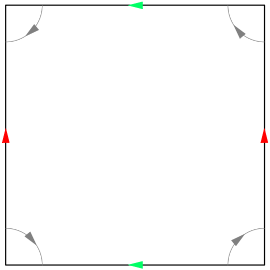

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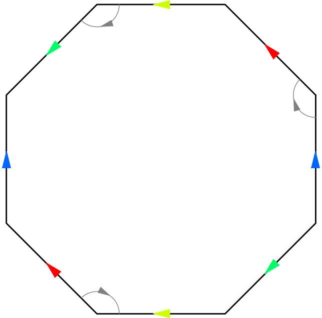

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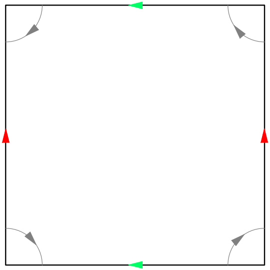

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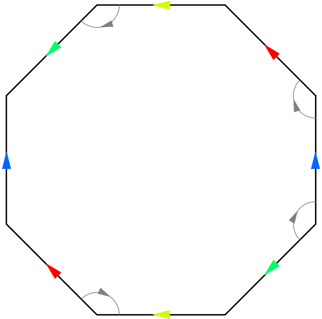

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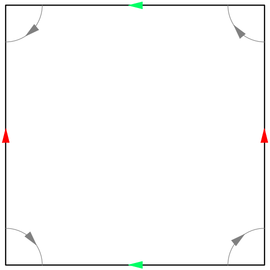

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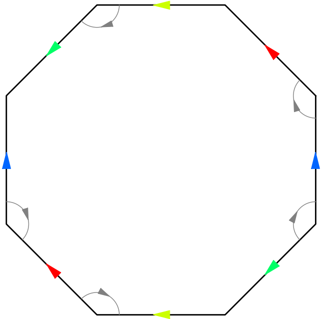

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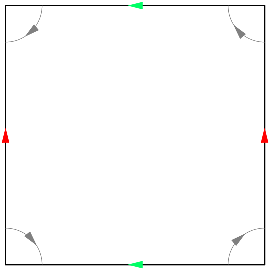

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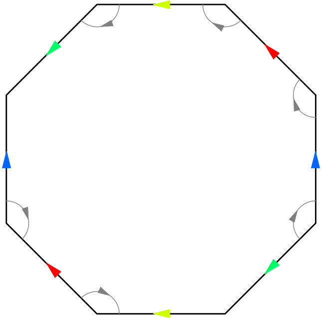

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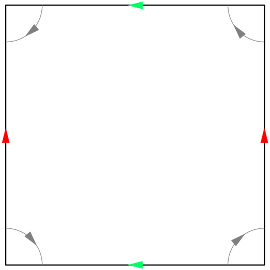

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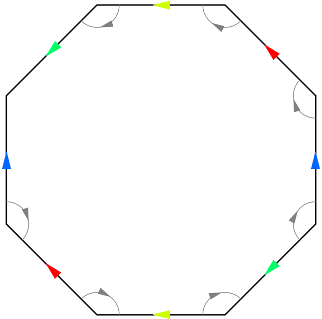

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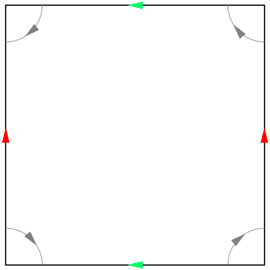

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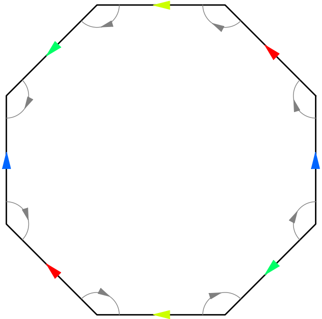
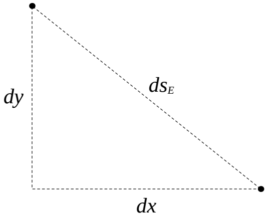


Outline

Geometry on the plane

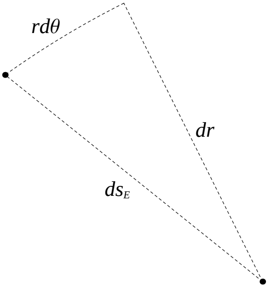
Cartesian coordinates:
Euclidean distance between (x, y) and $(x + dx, y + dy)$ is

$$ds_E^2 = dx^2 + dy^2$$



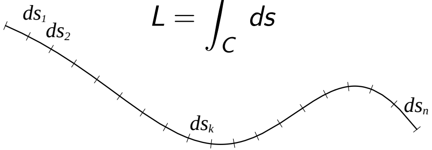
Polar coordinates:
Euclidean distance between (r, θ) and $(r + dr, \theta + d\theta)$ is

$$ds_E^2 = dr^2 + r^2 d\theta^2$$



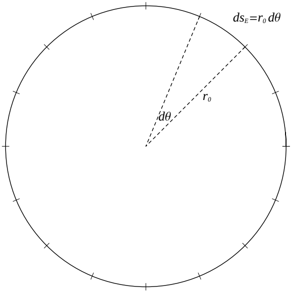
Geometry on the plane

- get length of a curve C by adding up (integrating) ds_E along curve:



$$L = \int_C ds$$

- Example: length of a circle of radius $r = r_0$



$$ds_E^2 = dr^2 + r^2 d\theta^2 = 0 + r_0^2 d\theta^2$$

so

$$L = \int_C ds = \int_0^{2\pi} r_0 \, d\theta = r_0 \cdot 2\pi$$

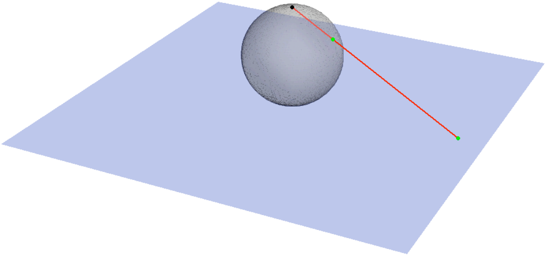
Geometry on the sphere

- Claim: With stereographic projection, lengths on the sphere are related to lengths in the plane by

$$ds_S^2 = \left(\frac{1}{1+\frac{1}{4}r^2}\right)^2 ds_E^2$$

- express ds_E in polar coordinates

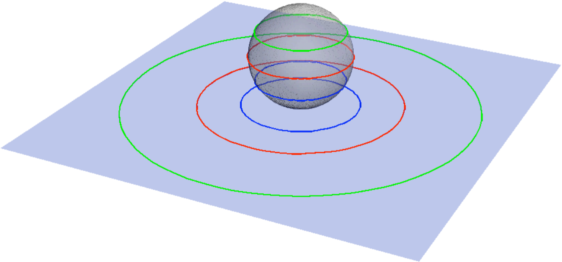
$$ds_S^2 = \left(\frac{1}{1+\frac{1}{4}r^2}\right)^2 (dr^2 + r^2 d\theta^2) = \left(\frac{1}{1+\frac{1}{4}r^2}\right)^2 dr^2 + \left(\frac{r}{1+\frac{1}{4}r^2}\right)^2 d\theta^2$$



Geometry on the sphere

- Example: spherical length of latitude circle
  - projects to a Euclidean circle of some radius $r = r_0$
  - again have $dr = 0$ so

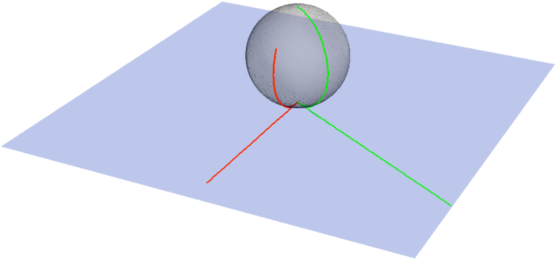
$$L_s = \int_C ds = \int_0^{2\pi} \frac{r_0}{1 + \frac{1}{4} r_0^2} \, d\theta = \frac{r_0}{1 + \frac{1}{4} r_0^2} \cdot 2\pi$$



Geometry on the sphere

- Example: spherical length of longitude semi-circle
  - projects to a Euclidean ray at some angle $\theta = \theta_0$
  - let r range from 0 to r_0
  - now have $d\theta = 0$ so

$$L_s = \int_0^{r_0} \frac{1}{1 + \frac{1}{4} r^2} \, dr = 2 \tan^{-1}\left(\frac{r_0}{2}\right)$$

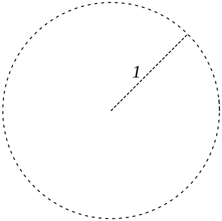


A different geometry

- a new distance expression

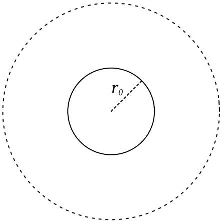
$$ds_H^2 = \left(\frac{1}{1-r^2}\right)^2 ds_E^2 = \left(\frac{1}{1-r^2}\right)^2 dr^2 + \left(\frac{r}{1-r^2}\right)^2 d\theta^2$$

- restrict to the Euclidean disk with $r < 1$



- H-length of Euclidean circle centered at origin
  - with $dr = 0$

$$L_H = \int_0^{2\pi} \frac{r_0}{1-r_0^2}\, d\theta = \frac{r_0}{1-r_0^2} \cdot 2\pi$$



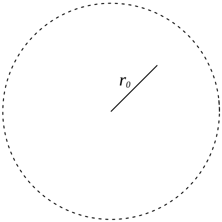
  - note that H-length increases without bound as r_0 approaches 1

A different geometry

- H-length of a Euclidean ray starting at origin
  - with $d\theta = 0$ and going from $r = 0$ to $r = r_0$

$$L_H = \int_0^{r_0} \frac{1}{1 - r^2}\, dr = \ln\left(\frac{1 + r_0}{1 - r_0}\right)$$

  - note that H-length increases without bound as r_0 approaches 1

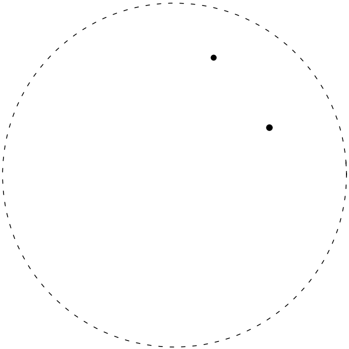


- H-length is short for "hyperbolic length"
- metric view of Poincaré disk model of the hyperbolic plane

Outline

Lines in the hyperbolic plane

- fix two points and ask "What path has the shortest H-distance between these two points?"
- shortest hyperbolic curve between two points is along the Euclidean circle through the points that is orthogonal to the boundary of the Euclidean disk $r < 1$
- refer to these shortest hyperbolic curves as *hyperbolic lines* or *H-lines*



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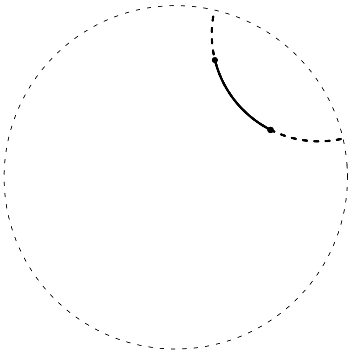

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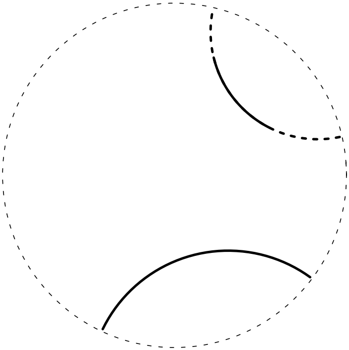

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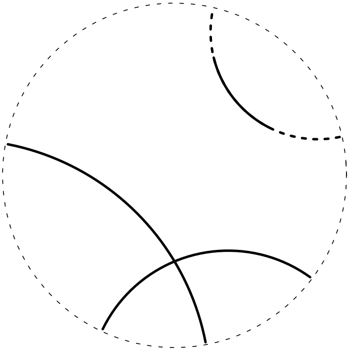
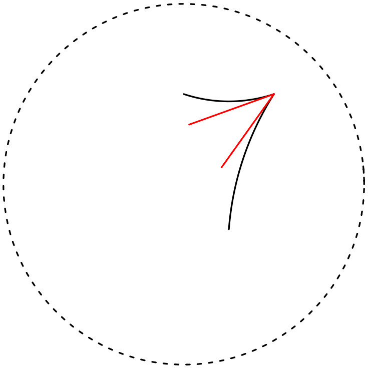


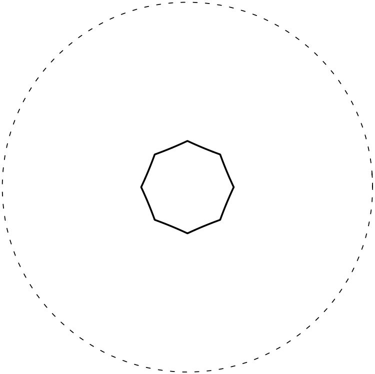
Angles in the hyperbolic plane

- hyperbolic angle between two hyperbolic lines with a common point is the Euclidean angle between the tangents to the Euclidean boundary-orthogonal circles



Octagons in the hyperbolic plane

- look at regular octagons in the hyperbolic plane
  - small octagon has interior angle of about $3\pi/4$
  - large octagon (vertices near boundary of disk) has interior angle of about 0
  - in between, there is a regular octagon with an interior angle of $\pi/4$.
  - can glue this octagon into a smooth "two-holed torus"



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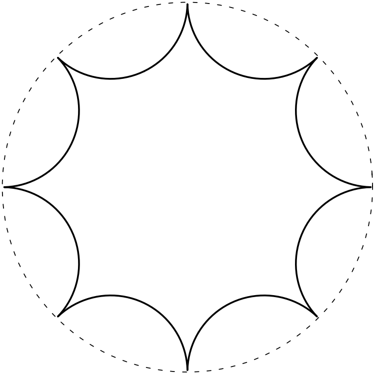

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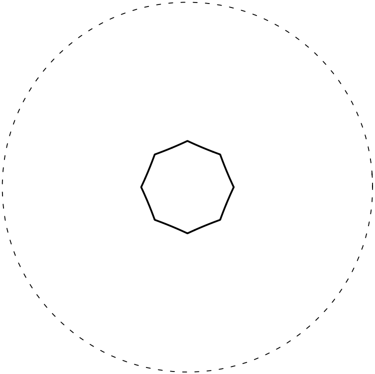

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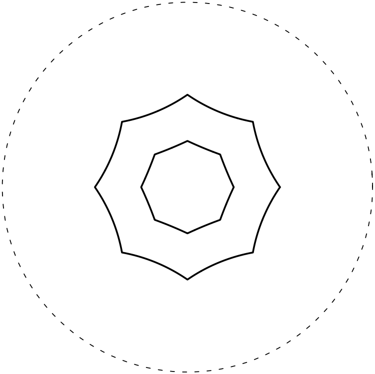

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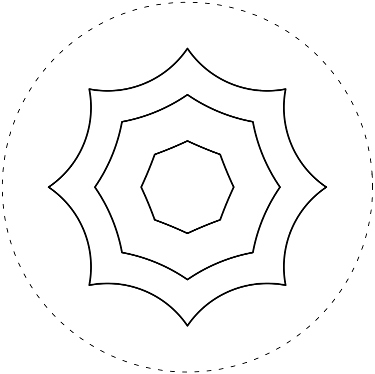

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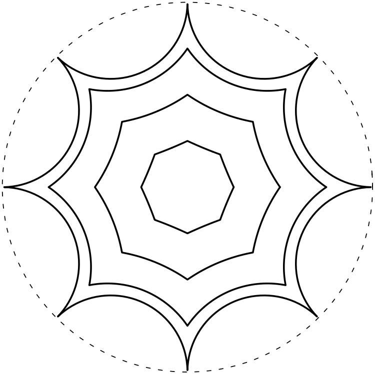
  - in between, there is a regular octagon with an interior angle of $\pi/4$.
  - can glue this octagon into a smooth "two-holed torus"



Outline

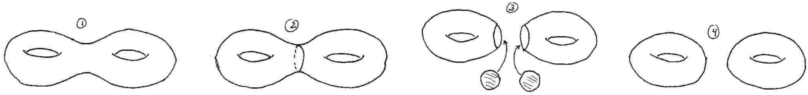
Geometrization of surfaces

- can construct a smooth "two-holed" torus using an octagon in the hyperbolic plane
- terminology: we say that the "two-holed" torus *admits a geometric structure modeled on the hyperbolic plane H^2*
- a geometry on a manifold is a *model geometry* if it is *simply connected* and *homogeneous*
  - simply connected: every loop can be shrunk to a point
  - homogeneous: the geometry of the manifold looks the same at all points
- for two dimension, there are three model geometries:
  - the "round" sphere S^2
  - the Euclidean plane E^2
  - the hyperbolic plane H^2
- model geometries provide a way of classifying closed two-dimensional manifolds
  - the sphere admits a geometric structure modeled on S^2
  - the torus admits a geometric structure modeled on E^2
  - for $n \geq 2$, an "n-holed" torus admits a geometric structure modeled on H^2

Outline

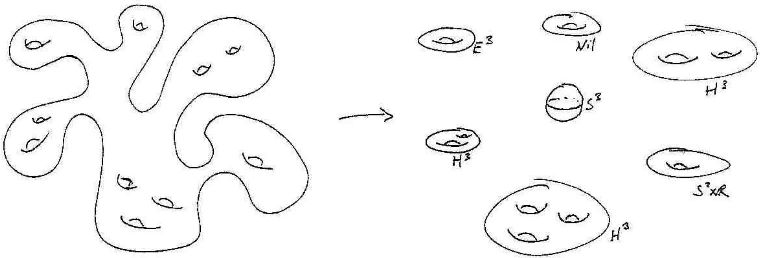
Geometrization of 3-manifolds

- does this classification by model geometries work in dimension three?
- in dimension three, there are 8 model geometries
  - three obvious generalizations: S^3, E^3, H^3 (these are homogeneous and isotropic)
  - five less obvious ones: $S^2 \times \mathbb{R}$, $H^2 \times \mathbb{R}$, $\widetilde{SL}(2, \mathbb{R})$, Nil, Sol (these are homogeneous but not isotropic)
- however, not every three-dimensional manifold admits a geometric structure
- to deal with this, do surgery
  - cut out any two-sphere that does not bound a solid ball
  - cut out any two-torus that does not bound a solid torus



The Geometrization Conjecture

- Theorem: A finite number of two-sphere and two-torus surgeries will decompose a closed three-manifold into pieces on which no further surgery is possible.
- Geometrization Conjecture: Any closed three-manifold can be decomposed into pieces by surgery and each piece admits a geometric structure based on one of the eight model geometries.



- proposed by William Thurston in 1982

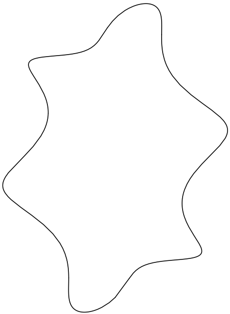
Back to the Poincaré Conjecture

- Geometrization Conjecture: Any closed three-manifold can be decomposed into pieces by surgery and each piece admits a geometric structure based on one of the eight model geometries.

- the Geometrization Conjecture implies the Poincaré Conjecture
  - simply connected implies no two-torus surgery possible
  - surgery by two-spheres produces closed pieces
  - only closed model geometry is S^3

Outline

Curvature of a plane curve



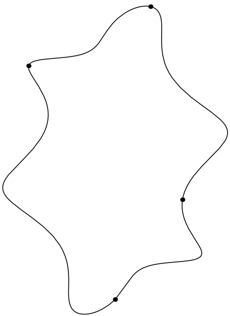
- at a point, find radius R of the "best-fit" circle
- define *curvature* as reciprocal of this radius:

$$k = \frac{1}{R}$$

- another view: rate at which the tangent vector changes with respect to distance along curve
  - pick origin and let $\vec{C}$ be position vector for point on curve
  - tangent vector is $\dfrac{d\vec{C}}{ds}$ where s is length along curve
  - curvature is rate at which tangent vector changes so

$$k = \left|\frac{d}{ds}\frac{d\vec{C}}{ds}\right| = \left|\frac{d^2\vec{C}}{ds^2}\right|$$

Curvature of a plane curve



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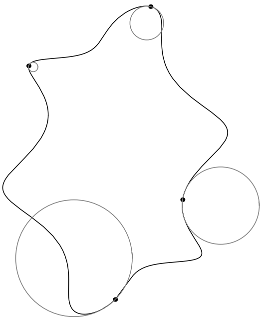

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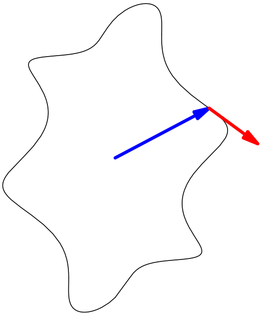

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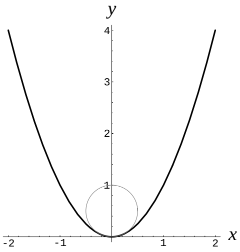
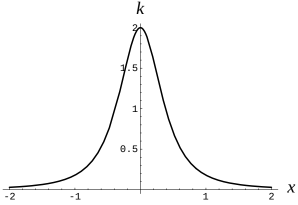
Curvature of a plane curve

- special case: curve is graph of a function $y = f(x)$
  - formula for curvature $$k(x) = \frac{\left|f''(x)\right|}{\left(1 + f'(x)^2\right)^{3/2}}$$
  - example: parabola $y = x^2$ has $$k(x) = \frac{2}{(1 + 4x^2)^{3/2}}$$





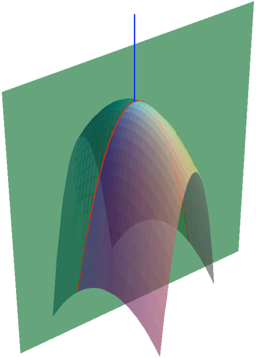
Outline

Curvature of a surface: extrinsic view

- at point P, cut surface with plane that contains the normal vector; intersecton of surface and plane is a curve in the plane; get curvature of this curve at P

- rotate plane around normal vector to look at curve curvatures of all cross-sections
- let k_1 be minimum curve curvature and k_2 be maximum curve curvature; define curvature of surface at the point P as product

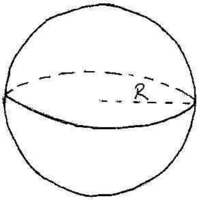
$$K = k_1 k_2$$



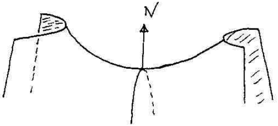
- called *Gaussian curvature* of the surface at point P

Curvature of a surface: extrinsic view

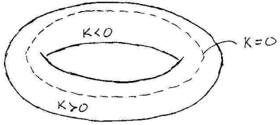
- example: round sphere of radius R has $K = \frac{1}{R^2}$ at all points



- example: saddle $z = x^2 - y^2$ has $K = (-2)(2) = -4$ at origin



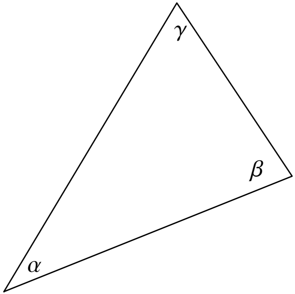
- example: "standard" torus has positive curvature on outer part and negative curvature on inner part



Curvature of a surface: intrinsic view

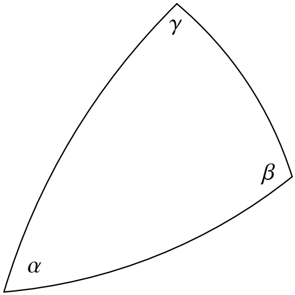
- let ΔT be a small triangle containing the point P; let α, β, and γ be radian measures of the three angles
- can check how much angle sum $\alpha + \beta + \gamma$ differs from π
- define curvature at P by

$$K = \lim_{\Delta T \to P} \frac{\pi - (\alpha + \beta + \gamma)}{\text{area of } \Delta T}$$

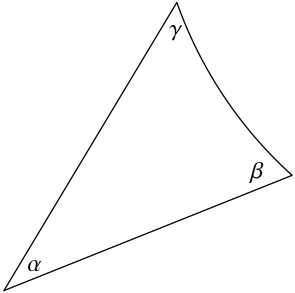


- example: triangles in Euclidean plane have $\alpha + \beta + \gamma = \pi$ so $K = 0$

Curvature of a surface: intrinsic view



- example: triangles in sphere have $\alpha + \beta + \gamma > \pi$ so $K > 0$



- example: triangles in hyperbolic plane have $\alpha + \beta + \gamma < \pi$ so $K < 0$

Outline

Geometry on manifolds

- geometry is encoded in the expression for ds
  - let (x_1, x_2) be some generic coordinates for the Euclidean plane
  - previous examples used $x_1 = r$ and $x_2 = \theta$
  - general expression for ds is

$$ds^2 = \sum_{i=1}^{2} \sum_{j=1}^{2} g_{ij}\, dx_i\, dx_j$$

  - think of the g_{ij} as entries in a 2×2 matrix g
  - example: for hyperbolic geometry

$$g_{11} = \left(\frac{1}{1-r^2}\right)^2 \qquad g_{22} = \left(\frac{r}{1-r^2}\right)^2 \qquad g_{12} = g_{21} = 0$$

$$g = \begin{bmatrix} \left(\frac{1}{1-r^2}\right)^2 & 0 \\ 0 & \left(\frac{r}{1-r^2}\right)^2 \end{bmatrix}$$

- g is called a *metric* and the g_{ij} are *components of the metric*
  - must be symmetric and positive definite (so $ds^2 \geq 0$)
  - easily generalize to higher dimensions by letting indices i and j range from 1 to n

Outline

Curvature of a manifold

- basic idea: at a point, use Gaussian curvatures of two-dimensional surfaces through that point; these are called *sectional curvatures*

- information on sectional curvatures encoded in *Riemann curvature tensor* Rm
  - think of Rm as machine that eats four vectors, spits out a number
  - $\text{Rm}(\vec{a}, \vec{b}, \vec{c}, \vec{d})$ is a number with specific geometric interpretation
  - special case: pick $\vec{u}$ and $\vec{v}$ to be perpendicular unit vectors
  - $\text{Rm}(\vec{u}, \vec{v}, \vec{u}, \vec{v})$ is the sectional curvature for the surface to which $\vec{u}$ and $\vec{v}$ are tangent

Curvature of a manifold

- there is a formula for components Rm_{ijkl} that involves second derivatives of the metric components g_{ij} with terms like

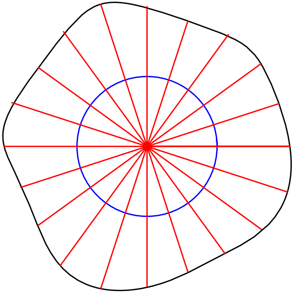
$$\frac{\partial^2 g_{ij}}{\partial x_k \partial x_l}$$

- two curvature quantities derived from Rm
  - let $\vec{e}_i$ be unit vector tangent to x_i coordinate direction
  - *Ricci curvature* Rc defined by $\mathrm{Rc}(\vec{a}, \vec{b}) = \sum_{i=1}^{n} \mathrm{Rm}(\vec{e}_i, \vec{a}, \vec{e}_i, \vec{b})$
  - *scalar curvature* R defined by $\mathrm{R} = \sum_{j=1}^{n} \mathrm{Rc}(\vec{e}_j, \vec{e}_j)$
  - think of these as averages of sectional curvatures

Outline

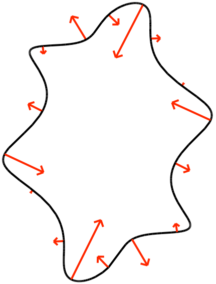
Curve shortening

- simple topology problem: is any closed curve in the plane (with no self-intersections) homeomorphic to a circle?
- easy answer for a convex curve: use radial projection



- need a complicated way to approach the problem for a more general closed curve; will look at a not-so-obvious idea

Curve shortening



- define a motion of the curve by assigning to each point a velocity that is perpendicular to the curve at that point with magnitude equal to the signed curvature

$$\frac{\partial \vec{C}}{\partial t} = k\vec{N}$$

- this is a partial differential equation in the category of heat equations
- analyze the initial value problem for this equation and find (Gage, Hamilton, Grayson, mid 80s)
  - for any initial curve, a solution exists
  - the length and area enclosed by the evolving curve decrease in time with area decreasing linearly
  - evolving curve becomes circular as area goes to zero
- can rescale to keep length constant in which case evolving curve converges to a circle as $t \to \infty$

Outline

Ricci flow

- analog of curve shortening for a general manifold
- Ricci flow defined by an evolution equation for the metric

$$\frac{\partial g_{ij}}{\partial t} = -2\,\mathrm{Rc}_{ij}$$

- sometimes convenient to rescale so volume of evolving geometry on manifold is constant; can do this by including an addition term

$$\frac{\partial g_{ij}}{\partial t} = -2\,\mathrm{Rc}_{ij} + \frac{2}{3}\bar{R} g_{ij}$$

  where $\bar{R}$ is the average of the scalar curvature over the manifold
- study initiated by Richard Hamilton with focus on dimension three
- fairly easy general results
  - any symmetry of the initial metric is preserved in the evolving geometry of solution
  - if the normalized flow converges, the limit is an *Einstein geometry* which are well understood

Outline

Ricci flow approach to geometrization

- first major result (Hamilton, 1982): if the initial metric has positive Ricci curvature at all points, then the (normalized) Ricci flow has a solution that converges to the geometry of the round three-sphere

- if Ricci flow does not converge, look at two cases depending on whether curvature remains bounded at all points or not
  - (Hamilton, 1999) if curvature remains bounded, manifold can be decomposed (with torus surgeries) into pieces that admit geometric structure
  - unbounded curvature at a point corresponds to the geometry ‘pinching down” in a singularity
  - Hamilton conjectured that these pinching singularities are related to two-sphere surgeries
  - idea: stop flow just before singularity, do surgery, restart flow on each resulting piece

Ricci flow approach to geometrization

- Hamilton's conjecture: For any initial metric on a three-manifold, Ricci flow with surgery with result in a finite number of pieces each of which admits a model geometry.
- work of Gregory Perelman
  - three preprints in 2002-2003
  - geometry near any singularity has standard structure on which surgery is possible
  - there are finitely many singularity formations
- within the last year, several groups have independently released more complete versions of a proof based on the work of Hamilton and Perelman
- Perelman offered a Fields Medal
- Science Magazine "Breakthrough of the Year" for 2006